TWMS J. Pure Appl. Math., V.7, N.1, 2016, pp.76-87

ON THE SPACES OF NÖRLUND NULL AND NÖRLUND CONVERGENT SEQUENCES

ORHAN TUĞ[1], FEYZİ BAŞAR[2]

Abstract. In this article, the sequence spaces $c_0(N^t)$ and $c(N^t)$ are introduced as the domain of Nörlund mean N^t in the spaces c_0 and c of null and convergent sequences which are isomorphic to the spaces c_0 and c, respectively, and some inclusion relations are given. Additionally, Schauder basis for the spaces $c_0(N^t)$ and $c(N^t)$ are constructed and their alpha-, beta- and gamma-duals are computed. Finally, the classes $(c(N^t) : \ell_\infty)$, $(c(N^t) : c)$ and $(c(N^t) : c_0)$ of matrix transformations are characterized.

Keywords: matrix domain, spaces of convergent and null sequences, Nörlund matrix, alpha-, beta- and gamma-duals and matrix transformations.

AMS Subject Classification: 46A45, 40C05.

1. Introduction

We denote the space of all complex valued sequences by ω. Each vector subspace of ω is called as a *sequence space*, as well. The spaces of all bounded, convergent and null sequences are denoted by ℓ_∞, c and c_0, respectively. By ϕ, we mean the space of all finitely non-zero sequences. A sequence space μ is called an FK-space if it is a complete linear metric space with continuous coordinates $p_n : \mu \to \mathbb{C}$ with $p_n(x) = x_n$ for all $x = (x_n) \in \mu$ and every $n \in \mathbb{N}$, where $\mathbb{C}$ denotes the complex field and $\mathbb{N} = \{0, 1, 2, \ldots\}$. A normed FK-spaces is called a BK-space, that is, a BK-space is a Banach space with continuous coordinates, [13, pp. 272-273]. The sequence spaces ℓ_∞, c and c_0 are BK-spaces with the usual sup-norm defined by $\|x\|_\infty = \sup_{k\in\mathbb{N}} |x_k|$. By ℓ_1, ℓ_p, cs, cs_0 and bs, we denote the spaces of all absolutely convergent, p-absolutely convergent, convergent, convergent to zero and bounded series, respectively; where $1 < p < \infty$.

The alpha-dual λ^α, beta-dual λ^β and gamma-dual λ^γ of a sequence space λ are defined by

$$\begin{aligned}
\lambda^\alpha &:= \{x = (x_k) \in \omega : xy = (x_k y_k) \in \ell_1 \text{ for all } y = (y_k) \in \lambda\},\\
\lambda^\beta &:= \{x = (x_k) \in \omega : xy = (x_k y_k) \in cs \text{ for all } y = (y_k) \in \lambda\},\\
\lambda^\gamma &:= \{x = (x_k) \in \omega : xy = (x_k y_k) \in bs \text{ for all } y = (y_k) \in \lambda\}.
\end{aligned}$$

Let λ, μ be any two sequence spaces and $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk}, where $k, n \in \mathbb{N}$. Then, we say that A defines a *matrix transformation* from λ into μ and we denote it by writing $A : \lambda \to \mu$, if for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = \{(Ax)_n\}$, the A-transform of x, is in μ; where

$$(Ax)_n = \sum_k a_{nk} x_k \tag{1}$$

[1]Department of Mathematics Education, Ishik University, Ishik Campus, Erbil, Iraq

[2]Department of Mathematics, Fatih University, The Hadımköy Campus, Büyükçekmece, Istanbul, Turkey
e-mail: orhan.tug@ishik.edu.iq, fbasar@fatih.edu.tr

Manuscript received March 2015.

provided the series on the right side of (1) converges for each $n \in \mathbb{N}$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By $(\lambda : \mu)$, we denote the class of all matrices A such that $A : \lambda \to \mu$. Thus, $A \in (\lambda : \mu)$ if and only if Ax exists, i.e. $A_n \in \lambda^\beta$ for all $n \in \mathbb{N}$ and belongs to μ for all $x \in \lambda$, where A_n denotes the sequence in the n-th row of A.

If a normed sequence space λ contains a sequence (b_n) with the following property that for every $x \in \lambda$ there is a unique sequence of scalars (α_n) such that

$$\lim_{n\to\infty} \|x - (\alpha_0 b_0 + \alpha_1 b_1 + \cdots + \alpha_n b_n)\| = 0$$

then (b_n) is called a Schauder basis for λ. The series $\sum_k \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n) and written as $x = \sum_k \alpha_k b_k$.

If λ is an FK-space, $\phi \subset \lambda$ and (e^k) is a basis for λ then λ is said to have AK property, where e^k is a sequence whose only term in k^{th} place is 1 the others are zero for each $k \in \mathbb{N}$ and $\phi = span\{e^k\}$. If ϕ is dense in λ, then λ is called AD-space, thus AK implies AD.

Let (t_k) be a nonnegative real sequence with $t_0 > 0$ and $T_n = \sum_{k=0}^n t_k$ for all $n \in \mathbb{N}$. Then, the Nörlund mean with respect to the sequence $t = (t_k)$ is defined by the matrix $N^t = (a^t_{nk})$ as follows

$$a^t_{nk} = \begin{cases} \frac{t_{n-k}}{T_n} & , \quad 0 \le k \le n, \\ 0 & , \quad k > n \end{cases} \tag{2}$$

for every $k, n \in \mathbb{N}$. It is known that the Nörlund matrix N^t is regular if and only if $t_n/T_n \to 0$, as $n \to \infty$ ([18], Theorem 16, p. 64), and is reduced in the case $t = e = (1, 1, 1, \ldots)$ to the matrix C_1 of arithmetic mean. Additionally, for $t_n = A^{r-1}_n$ for all $n \in \mathbb{N}$, the method N^t is reduced to the Cesàro method C_r of order $r > -1$, where

$$A^r_n = \begin{cases} \frac{(r+1)(r+2)\cdots(r+n)}{n!} & , \quad n = 1, 2, 3, \ldots, \\ 1 & , \quad n = 0. \end{cases}$$

Let $t_0 = D_0 = 1$ and define D_n for $n \in \{1, 2, 3, \ldots\}$ by

$$D_n = \begin{vmatrix} t_1 & 1 & 0 & 0 & \cdots & 0 \\ t_2 & t_1 & 1 & 0 & \cdots & 0 \\ t_3 & t_2 & t_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{n-1} & t_{n-2} & t_{n-3} & t_{n-4} & \cdots & 1 \\ t_n & t_{n-1} & t_{n-2} & t_{n-3} & \cdots & t_1 \end{vmatrix}. \tag{3}$$

Then, the inverse matrix $U^t = (u^t_{nk})$ of Nörlund matrix N^t was defined by Mears in [27] for all $n \in \mathbb{N}$, as follows;

$$u^t_{nk} = \begin{cases} (-1)^{n-k} D_{n-k} T_k & , \quad 0 \le k \le n, \\ 0 & , \quad k > n. \end{cases} \tag{4}$$

Additionally, the inverse of Nörlund matrix and some multiplication theorems for Nörlund mean were studied by Mears [26, 27].

The domain λ_A of an infinite matrix A in a sequence space λ is defined by

$$\lambda_A = \{x = (x_k) \in \omega : Ax \in \lambda\}$$

which is a sequence space. The domain of Nörlund matrix N^t in the classical sequence spaces ℓ_∞ and ℓ_p were introduced by Wang [31], where $1 \le p < \infty$. We should note here that as a new development, the reader may refer to [14] for studying on sequence spaces and related topics in the sense of multiplicative calculus.

The rest of this paper is organized, as follows:

In section 2, we introduce the sequence spaces $c_0(N^t)$ and $c(N^t)$, and give their some algebraic and topological properties. Section 3 is devoted to the determination of the alpha-, beta- and gamma-duals of the spaces $c_0(N^t)$ and $c(N^t)$. In Section 4, the classes $(c(N^t) : \ell_\infty)$, $(c(N^t) : c)$ and $(c(N^t) : c_0)$ of matrix transformations are characterized and the characterizations of some other classes are also derived as an application of those main results. In the final section of the paper, we note the significance of the present results in the literature related with the domain of certain triangle matrices on the spaces c_0 and c, and record some further suggestions.

2. The sequence spaces $c_0(N^t)$ and $c(N^t)$ of non-absolute type

We introduce the sequence spaces $c_0(N^t)$ and $c(N^t)$ as the set of all sequences whose N^t-transforms are in the spaces of null and convergent sequences, respectively, that is

$$\begin{aligned} c_0(N^t) &:= \left\{ x = (x_k) \in \omega : \lim_{n\to\infty} \frac{1}{T_n} \sum_{k=0}^{n} t_{n-k} x_k = 0 \right\}, \\ c(N^t) &:= \left\{ x = (x_k) \in \omega : \exists l \in \mathbb{C} \text{ such that } \lim_{n\to\infty} \frac{1}{T_n} \sum_{k=0}^{n} t_{n-k} x_k = l \right\}. \end{aligned}$$

We define the sequence $y = (y_k)$ by the N^t-transform of a sequence $x = (x_k)$, that is,

$$y_k = (N^t x)_k = \frac{1}{T_k} \sum_{j=0}^{k} t_{k-j} x_j \tag{5}$$

for all $k \in \mathbb{N}$. Therefore, by applying U^t to the sequence y defined by (5) we obtain that

$$x_k = (U^t y)_k = \sum_{j=0}^{k} (-1)^{k-j} D_{k-j} T_j y_j \tag{6}$$

for all $k \in \mathbb{N}$. Throughout the text, we suppose that the terms of the sequences $x = (x_k)$ and $y = (y_k)$ are connected with the relation (5).

Theorem 2.1. *The sequence spaces $c_0(N^t)$ and $c(N^t)$ are the linear spaces with the co-ordinatewise addition and scalar multiplication which are the BK-spaces with the norm $\|x\|_{c_0(N^t)} = \|x\|_{c(N^t)} = \|N^t x\|_\infty$.*

Proof. The proof of the first part of the theorem is a routine verification and so, we omit the detail.

Since c_0 and c are the BK-spaces with respect to their usual sup-norm and N^t is a triangle matrix, Theorem 4.3.2 of Wilansky [32, p. 61] gives the fact that $c_0(N^t)$ and $c(N^t)$ are the BK-spaces. This completes the proof. □

Let λ denotes any of the spaces c_0 or c. With the notation of (5), since the transformation $T : \lambda(N^t) \to \lambda$ defined by $x \mapsto y = Tx = N^t x$ is a norm preserving linear bijection, we have the following:

Corollary 2.2. *The sequence space $\lambda(N^t)$ is linearly norm isomorphic to the space λ, where $\lambda \in \{c_0, c\}$.*

Theorem 2.3. *Let N^t be a non-Mercerian matrix, i.e., $c_{N^t} \neq c$. Then, the inclusions $c_0 \subset c_0(N^t)$ and $c \subset c(N^t)$ strictly hold.*

Proof. Suppose that N^t is a non-Mercerian matrix. To show the inclusion relation $c_0 \subset c_0(N^t)$ holds we take any sequence $y \in c_0$. Then, by using the regularity property of N^t we can easily find that $N^t y \in c_0$ which means that $y \in c_0(N^t)$. That is to say that the inclusion $c_0 \subset c_0(N^t)$ holds. In the similar way, it is trivial to see that the inclusion $c \subset c(N^t)$ also holds.

To prove the second part of the theorem, we should show that the sets $c_0(N^t) - c_0$ and $c(N^t) - c$ are not empty. For this, consider the sequence $v = (v_k) = \{(-1)^k\}$ which does not belong to both of the spaces c_0 and c. Since

$$\lim_{k\to\infty} (C_1 v)_k = \lim_{k\to\infty} \frac{1}{k+1} \sum_{j=0}^{k} (-1)^j = 0$$

and Cesàro mean C_1 of order one is a special case of Nörlund mean, we have $v \in c_0(N^t)$. Additionally, since $C_1 v \in c_0$ implies $C_1 v \in c$; we also have $v \in c(N^t)$. Hence, $v \in \left[c_0(N^t) - c_0\right] \cap \left[c(N^t) - c\right]$. That is to say that the inclusions $c_0 \subset c_0(N^t)$ and $c \subset c(N^t)$ strictly hold. $\square$

It is known from Theorem 2.3 of Jarrah and Malkowsky [19] that the domain λ_T of an infinite matrix $T = (t_{nk})$ in a normed sequence space λ has a basis if and only if λ has a basis, if T is a triangle. As a direct consequence of this fact, we have:

Corollary 2.4. *Let $\alpha_k = (N^t x)_k$ for all $k \in \mathbb{N}$. Define the sequence $\{u^{(n)}\} = \left\{u_k^{(n)}\right\}_{k\in\mathbb{N}}$ in the space $c_0(N^t)$ by*

$$u_k^{(n)} = \begin{cases} (-1)^{n-k} D_{n-k} T_k & , \quad 0 \leq k \leq n, \\ 0 & , \quad k > n \end{cases} \tag{7}$$

for every fixed $n \in \mathbb{N}$.

(a) *The sequence $\{u^{(n)}\}_{n\in\mathbb{N}}$ is a basis for the space $c_0(N^t)$ and any $x \in c_0(N^t)$ has a unique representation of the form $x = \sum_{k=0}^{\infty} \alpha_k u_k^n$.*

(b) *The set $\{e, u^{(n)}\}$ is a basis for the sequence space $c(N^t)$ and any $x \in c(N^t)$ has a unique representation of the form $x = le + \sum_{k=0}^{\infty} (\alpha_k - l) u_k^n$, where $l = \lim_{k\to\infty} \alpha_k$.*

3. The alpha-, beta- and gamma-duals of the spaces $c_0(N^t)$ and $c(N^t)$

In this section, the alpha-, beta- and gamma-duals of the spaces $c_0(N^t)$ and $c(N^t)$ are determined.

Now, we start with the following lemma which is needed in proving our theorems. Here and after, we denote the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{F}$.

Lemma 3.1. *Let $A = (a_{nk})$ be an infinite matrix over the complex field. Then, the following statements hold:*

(a) *$A \in (c_0 : \ell_1) = (c : \ell_1)$ if and only if*

$$\sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K} a_{nk} \right| < \infty. \tag{8}$$

(b) *$A \in (c : \ell_\infty)$ if and only if*

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}| < \infty. \tag{9}$$

(c) $A \in (c : c)$ *if and only if (9) holds, and*

$$\exists a_k \in \mathbb{C} \quad \text{such that} \quad \lim_{n\to\infty} a_{nk} = a_k \quad \text{for all } k \in \mathbb{N}, \tag{10}$$

$$\exists a \in \mathbb{C} \quad \text{such that} \quad \lim_{n\to\infty} \sum_k a_{nk} = a. \tag{11}$$

(d) $A \in (c : c_0)$ *if and only if (9) holds, and*

$$\lim_{n\to\infty} a_{nk} = 0 \quad \text{for all } k \in \mathbb{N}, \tag{12}$$

$$\lim_{n\to\infty} \sum_k a_{nk} = 0. \tag{13}$$

Theorem 3.2. *The α-dual of the spaces $c_0(N^t)$ and $c(N^t)$ is the set*

$$d_1^t := \left\{ a = a_k \in \omega : \sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K} (-1)^{n-k} D_{n-k} T_k a_n \right| < \infty \right\}.$$

Proof. Let us define the matrix $B = (b_{nk}^t)$ with the aid of $a = (a_k) \in \omega$ by

$$b_{nk}^t = \begin{cases} (-1)^{n-k} D_{n-k} T_k a_n & , \quad 0 \le k \le n \\ 0 & , \quad k > n \end{cases}$$

for all $k, n \in \mathbb{N}$. Since the relation (6) holds, we easily obtain that

$$a_n x_n = \sum_{k=0}^{n} (-1)^{n-k} D_{n-k} T_k a_n y_k = (By)_n \tag{14}$$

for all $n \in \mathbb{N}$. From (14), we conclude that $ax = (a_n x_n) \in \ell_1$ whenever $x \in c_0(N^t)$ or $\in c(N^t)$ if and only if $By \in \ell_1$ whenever $y \in c_0$ or $\in c$. Therefore, we derive by Part (a) of Lemma 3.1 that

$$\sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K} (-1)^{n-k} D_{n-k} T_k a_n \right| < \infty$$

which leads to the desired result that $\{c_0(N^t)\}^\alpha = \{c(N^t)\}^\alpha = d_1^t$. $\square$

Theorem 3.3. *Define the set d_2^t, as follows;*

$$d_2^t \quad := \quad \left\{ a = (a_k) \in \omega : \sup_{n\in\mathbb{N}} \sum_k \left| \sum_{j=k}^{n} (-1)^{j-k} D_{j-k} T_k a_j \right| < \infty \right\}.$$

Then, $\{c_0(N^t)\}^\beta = \{c(N^t)\}^\beta = d_2^t \cap cs$.

Proof. Let $x = (x_k)$ be in $c_0(N^t)$ or $c(N^t)$. Now, consider the equality

$$\sum_{k=0}^{n} a_k x_k \quad = \quad \sum_{k=0}^{n-1} \sum_{j=k}^{n} (-1)^{j-k} a_j D_{j-k} T_k y_k + a_n T_n y_n = (Ey)_n \quad \text{for all} \quad n \in \mathbb{N}, \tag{15}$$

where $E = (e_{nk})$ is defined by

$$e_{nk} = \begin{cases} \sum_{j=k}^{n} (-1)^{j-k} D_{j-k} T_k a_j & , \quad 0 \le k \le n-1, \\ a_n T_n & , \quad k = n, \\ 0 & , \quad k > n \end{cases}$$

for all $k, n \in \mathbb{N}$. Then, we observe by taking into the equality (15) that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in c(N^t)$ if and only if $Ey \in c$ whenever $y = (y_k) \in c$. This is equivalent to the statement that "$a = (a_k) \in \{c(N^t)\}^\beta$ if and only if $E \in (c : c)$". Therefore, we derive from (15) and Part (c) of Lemma 3.1 that the sequence (a_k) satisfies the following conditions, respectively,

$$\sup_{n\in\mathbb{N}}\sum_k\left|\sum_{j=k}^{n}(-1)^{j-k}D_{j-k}T_ka_j\right|<\infty.$$
$$(a_k)\in cs.$$

This shows that $\{c(N^t)\}^\beta=d_2^t\cap cs$, as asserted. □

Theorem 3.4. *The γ-dual of the spaces $c_0(N^t)$ and $c(N^t)$ is the set d_2^t.*

Proof. This is similar to the proof of Theorem 3.3 with Part (b) of Lemma 3.1 instead of Part (c) of Lemma 3.1. So, we omit the detail. □

4. Matrix transformations related to the sequence space $c(N^t)$

In this section, we characterize some matrix classes from the spaces $c(N^t)$ into the classical sequence spaces ℓ_∞, c and c_0. Additionally, we characterize the class of matrix transformations from a given sequence space μ to the space $c(N^t)$.

Throughout this section, we define the matrices $F=(f_{nk})$ and $G=(g_{nk})$ via multiplication of the matrices A and N^t by the products AN^t and N^tA, respectively, that is

$$f_{nk}:=\sum_{j=k}^{\infty}(-1)^{j-k}D_{j-k}T_ka_{nj}\quad\text{and}\quad g_{nk}:=\sum_{j=0}^{n}\frac{t_{n-j}}{T_n}a_{jk}$$

for all $k,n\in\mathbb{N}$.

Theorem 4.1. *$A=(a_{nk})\in(c(N^t):\ell_\infty)$ if and only if*

$$A_n\in\{c(N^t)\}^\beta\quad\textit{for each}\quad n\in\mathbb{N},\tag{16}$$
$$F\in(c:\ell_\infty).\tag{17}$$

Proof. Suppose that $A=(a_{nk})\in(c(N^t):\ell_\infty)$ and $x=(x_k)\in c(N^t)$. Consider the following equality derived from the m^{th} partial sum of the series $\sum_k a_{nk}x_k$:

$$\sum_{k=0}^{m}a_{nk}x_k=\sum_{k=0}^{m}\sum_{j=k}^{m}(-1)^{j-k}D_{j-k}a_{nj}T_ky_k\tag{18}$$

for all $m,n\in\mathbb{N}$. Since Ax exists and belongs to the space ℓ_∞, the necessity of the condition (16) is obvious. Therefore, by letting $m\to\infty$ in the equality (18) one can see that

$$\sum_k a_{nk}x_k=\sum_k\sum_{j=k}^{\infty}(-1)^{j-k}D_{j-k}T_ka_{nj}y_k\tag{19}$$

for all $n\in\mathbb{N}$, i.e., $Ax=Fy$ which gives that $Fy\in\ell_\infty$. That is to say that $F\in(c:\ell_\infty)$.

Conversely, let us suppose that the conditions (16) and (17) hold, and take $x=(x_k)\in c(N^t)$. Then, (16) implies the existence of Ax and since the spaces $c(N^t)$ and c are isomorphic we have

$y \in c$. Therefore, (19) gives with (9) with f_{nk} instead of a_{nk} that

$$\begin{aligned}
\|Ax\|_\infty &= \sup_{n\in\mathbb{N}} \left| \sum_k a_{nk} x_k \right| \\
&\leq \sup_{n\in\mathbb{N}} \sum_k \left| \sum_{j=k}^{\infty} (-1)^{j-k} D_{j-k} T_k a_{nj} y_k \right| \\
&\leq \|y\|_\infty \left[\sup_{n\in\mathbb{N}} \sum_k \left| \sum_{j=k}^{\infty} (-1)^{j-k} D_{j-k} T_k a_{nj} \right| \right] < \infty.
\end{aligned}$$

Hence, $A \in (c(N^t) : \ell_\infty)$.

This completes the proof. $\square$

Theorem 4.2. *$A = (a_{nk}) \in (c(N^t) : c)$ if and only if the condition (16) holds, and*

$$F \in (c : c). \tag{20}$$

Proof. Suppose that the conditions (16) and (20) hold, and take any $x = (x_k) \in c(N^t)$. The condition (16) implies the existence of A-transform of x. Therefore, one can derive by using the hypothesis (9) with f_{nk} instead of a_{nk} that

$$\sum_{k=0}^{m} |a_k| \leq \sup_{n\in\mathbb{N}} \sum_k |f_{nk}| < \infty$$

for all $m \in \mathbb{N}$. Hence, $(a_k) \in \ell_1$ which implies that $(a_k y_k) \in \ell_1$. Then, we derive by letting $n \to \infty$ on (19) with (9) with f_{nk} instead of a_{nk} that

$$\lim_{n\to\infty} (Ax)_n = \lim_{n\to\infty} \sum_k f_{nk} y_k = \sum_k a_k y_k. \tag{21}$$

Since $(a_k y_k) \in \ell_1$, (21) gives that $Ax \in c$, that is, $A \in (c(N^t) : c)$.

Conversely, suppose that $A = (a_{nk}) \in (c(N^t) : c)$ and take $x \in c(N^t)$. Since the inclusion relation $c \subset \ell_\infty$ holds, the necessity of the conditions (16) and (9) with f_{nk} instead of a_{nk} follows from Theorem 4.1.

Now, consider the convergent sequences $u = (u_k) = \left\{ u_k^{(n)} \right\}_{k\in\mathbb{N}}$ defined by (7) and $x = (x_k) = \left\{ \sum_{j=0}^{k} (-1)^{k-j} D_{k-j} T_j \right\}$. Since A-transforms of u and x exist and belong to the space c by the hypothesis, one can see that $Au = \left\{ \sum_{j=k}^{\infty} (-1)^{j-k} D_{j-k} T_k a_{nj} \right\}_{n\in\mathbb{N}} \in c$ and $Ax = \left(\sum_k f_{nk} \right)_{n\in\mathbb{N}} \in c$ which shows the necessity of the conditions (9) and (11) with f_{nk} instead of a_{nk}, respectively. Hence, $F \in (c : c)$.

This completes the proof. $\square$

Corollary 4.3. *$A = (a_{nk}) \in (c(N^t) : c_0)$ if and only if (16) holds and (12) and (13) also hold with f_{nk} instead of a_{nk}, respectively.*

Now, we can give the theorem characterizing the class of matrix transformations from a given sequence space μ to the Nörlund space $c(N^t)$.

Theorem 4.4. *Suppose that μ be any given sequence space. Then, $A \in (\mu : c(N^t))$ if and only if $G \in (\mu : c)$.*

Proof. Let $x = (x_k) \in \mu$. Consider the following equality

$$\sum_{j=0}^{n} \frac{t_{n-j}}{T_n} \sum_{k=0}^{m} a_{jk} x_k = \sum_{k=0}^{m} g_{nk} x_k \text{ for all } m, n \in \mathbb{N}. \tag{22}$$

Then, by letting $m \to \infty$ in (22) one can see that $\{N^t(Ax)\}_n = (Gx)_n$ for all $n \in \mathbb{N}$. Since $Ax \in c(N^t)$, $N^t(Ax) = Gx \in c$. This completes the proof. □

Let $0 < r < 1$, $q = (q_k)$ be a sequence of non-negative reals with $q_0 > 0$ and $Q_n = \sum_{k=0}^{n} q_k$ for all $n \in \mathbb{N}$. Let us define the summation matrix $S = (s_{nk})$, the backward difference matrix $\Delta = (d_{nk})$, the Riesz matrix $R^q = (r_{nk}^q)$ with respect to the sequence $q = (q_k)$, the matrix $A^r = (a_{nk}^r)$ and the matrix $E^r = (e_{nk}^r)$ of Euler mean of order r by

$$s_{nk} := \begin{cases} 1 & , \ 0 \le k \le n, \\ 0 & , \ k > n, \end{cases} \qquad d_{nk} := \begin{cases} (-1)^{n-k} & , \ n-1 \le k \le n, \\ 0 & , \ \le k < n-1 \text{ or } k > n, \end{cases}$$

$$r_{nk}^q = \begin{cases} \frac{q_k}{Q_n} & , \ 0 \le k \le n, \\ 0 & , \ k > n, \end{cases} \qquad a_{nk}^r := \begin{cases} \frac{1+r^k}{n+1} u_k & , \ 0 \le k \le n, \\ 0 & , \ k > n, \end{cases}$$

$$e_{nk}^r := \begin{cases} \binom{n}{k} (1-r)^{n-k} r^k & , \ 0 \le k \le n, \\ 0 & , \ k > n, \end{cases}$$

for all $k, n \in \mathbb{N}$.

By combining Theorems 4.1, 4.2 and Corollary 4.3 with Theorem 4.4, the following results are derived on the characterization of some matrix classes concerning with the space $c(N^t)$ of Nörlund convergent sequences:

Corollary 4.5. *Let $A = (a_{nk})$ be an infinite matrix over the complex field. Then, the following statements hold:*

(i) *$A \in (c(N^t) : bs)$ if and only if (16) holds and (17) also holds with SF instead of F.*

(ii) *$A \in (c(N^t) : bv_\infty)$ if and only if (16) holds and (17) also holds with ΔF instead of F; where bv_∞ denotes the space of all sequences $x = (x_k)$ such that $(x_k - x_{k-1}) \in \ell_\infty$,* (cf. Başar and Altay [8]).

(iii) *$A \in (c(N^t) : X_\infty)$ if and only if (16) holds and (17) also holds with $C_1 F$ instead of F; where X_∞ denotes the space of all sequences $x = (x_k)$ such that $\left(\sum_{k=0}^{n} \frac{1}{n+1} x_k\right) \in \ell_\infty$, (cf. Ng and Lee [28]).*

(iv) *$A \in (c(N^t) : r_\infty^q)$ if and only if (16) holds and (17) also holds with $R^q F$ instead of F; where r_∞^q denotes the space of all sequences $x = (x_k)$ such that $\left(\sum_{k=0}^{n} \frac{q_k}{Q_n} x_k\right) \in \ell_\infty$,* (cf. Altay and Başar [1]).

(v) *$A \in (c(N^t) : a_\infty^r)$ if and only if (16) holds and (17) also holds with $A^r F$ instead of F; where a_∞^r denotes the space of all sequences $x = (x_k)$ such that $\left(\sum_{k=0}^{n} \frac{1+r^k}{1+n} x_k\right) \in \ell_\infty$,* (cf. Aydın and Başar [6]).

(vi) *$A \in (c(N^t) : e_\infty^r)$ if and only if (16) holds and (17) also holds with $E^r F$ instead of F; where e_∞^r denotes the space of all sequences $x = (x_k)$ such that $\left\{\sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k x_k\right\} \in \ell_\infty$,* (cf. Altay et al. [2]).

Corollary 4.6. *Let $A = (a_{nk})$ be an infinite matrix over the complex field. Then, the following statements hold:*

(i) $A \in (c(N^t) : cs)$ *if and only if (16) holds and (20) also holds with SF instead of F.*

(ii) $A \in (c(N^t) : c(\Delta))$ *if and only if (16) holds and (20) also holds with ΔF instead of F; where $c(\Delta)$ denotes the space of all sequences $x = (x_k)$ such that $(x_k - x_{k-1}) \in c$,* (cf. Başar [9]).

(iii) $A \in (c(N^t) : \widetilde{c})$ *if and only if (16) holds and (20) also holds with C_1F instead of F; where $\widetilde{c}$ denotes the space of all sequences $x = (x_k)$ such that $\left(\sum_{k=0}^{n} \frac{1}{n+1} x_k\right) \in c$,* (cf. Şengönül and Başar [30]).

(iv) $A \in (c(N^t) : r_c^q)$ *if and only if (16) holds and (20) also holds with R^qF instead of F; where r_c^q denotes the space of all sequences $x = (x_k)$ such that $\left(\sum_{k=0}^{n} \frac{q_k}{Q_n} x_k\right) \in c$,* (cf. Altay and Başar [1]).

(v) $A \in (c(N^t) : a_c^r)$ *if and only if (16) holds and (20) also holds with A^rF instead of F; where a_c^r denotes the space of all sequences $x = (x_k)$ such that $\left(\sum_{k=0}^{n} \frac{1+r^k}{1+n} x_k\right) \in c$,* (cf. Aydın and Başar [5]).

(vi) $A \in (c(N^t) : e_c^r)$ *if and only if (16) holds and (20) also holds with E^rF instead of F; where e_c^r denotes the space of all sequences $x = (x_k)$ such that $\left\{\sum_{k=0}^{n} \binom{n}{k}(1-r)^{n-k} r^k x_k\right\} \in c$,* (cf. Altay and Başar [4]).

Corollary 4.7. *Let $A = (a_{nk})$ be an infinite matrix over the complex field. Then, the following statements hold:*

(i) $A \in (c(N^t) : cs_0)$ *if and only if the condition (16) holds and the conditions (11), (12) and (13) also hold with $(SF)_{nk}$ instead of a_{nk} for all $k, n \in \mathbb{N}$, where cs_0 denotes the space of all series converging to zero.*

(ii) $A \in (c(N^t) : c_0(\Delta))$ *if and only if the condition (16) holds and the conditions (11), (12) and (13) also hold with $(\Delta F)_{nk}$ instead of a_{nk} for all $k, n \in \mathbb{N}$; where $c_0(\Delta)$ denotes the space of all sequences $x = (x_k)$ such that $(x_k - x_{k-1}) \in c_0$,* (cf. Başar [9]).

(iii) $A \in (c(N^t) : \widetilde{c}_0)$ *if and only if the condition (16) holds and the conditions (11), (12) and (13) also hold with $(C_1F)_{nk}$ instead of a_{nk} for all $k, n \in \mathbb{N}$; where $\widetilde{c}_0$ denotes the space of all sequences $x = (x_k)$ such that $\left(\sum_{k=0}^{n} \frac{1}{n+1} x_k\right) \in c_0$,* (cf. Şengönül and Başar [30]).

(iv) $A \in (c(N^t) : r_0^q)$ *if and only if the condition (16) holds and the conditions (11), (12) and (13) also hold with $(R^qF)_{nk}$ instead of a_{nk} for all $k, n \in \mathbb{N}$; r_0^q denotes the space of all sequences $x = (x_k)$ such that $\left(\sum_{k=0}^{n} \frac{q_k}{Q_n} x_k\right) \in c_0$,* (cf. Altay and Başar [1]).

(v) $A \in (c(N^t) : a_0^r)$ *if and only if the condition (16) holds and the conditions (11), (12) and (13) also hold with $(A^rF)_{nk}$ instead of a_{nk} for all $k, n \in \mathbb{N}$; where a_0^r denotes the space of all sequences $x = (x_k)$ such that $\left(\sum_{k=0}^{n} \frac{1+r^k}{1+n} x_k\right) \in c_0$,* (cf. Aydın and Başar [5]).

(vi) $A \in (c(N^t) : e_0^r)$ *if and only if the condition (16) holds and the conditions (11), (12) and (13) also hold with $(E^rF)_{nk}$ instead of a_{nk} for all $k, n \in \mathbb{N}$; where e_0^r denotes the space of all sequences $x = (x_k)$ such that $\left\{\sum_{k=0}^{n} \binom{n}{k}(1-r)^{n-k} r^k x_k\right\} \in c_0$,* (cf. Altay and Başar [4]).

5. Conclusion

In 1978, the domain of Nörlund matrix N^t in the classical sequence spaces ℓ_∞ and ℓ_p were introduced by Wang [31], where $1 \leq p < \infty$. In 1978, the domain of Cesàro matrix C_1 of order one in the classical sequence spaces ℓ_∞ and ℓ_p were introduced by Ng and Lee [28], where $1 \leq p < \infty$. Following Ng and Lee [28], Şengönül and Başar [30] have studied the domain of Cesàro matrix C_1 of order one in the classical sequence spaces c_0 and c. Following Şengönül and Başar [30], to fill up the gap in the existing literature we have worked on the domain of Nörlund matrix N^t in the classical sequence spaces c_0 and c.

Although the matrix transformations from the domain of certain triangles in the classical sequence spaces into the classical sequence spaces have been characterized, the matrix transformations from the domain of Nörlund matrix in the spaces of null and convergent sequences into some classical sequence spaces have not been characterized, until now. So, Theorems 4.1, 4.2, 4.4 and Corollary 4.3 have a special importance for this type studies, in future.

To review the concerning literature about the domain of the infinite matrix A in the sequence spaces c_0 and c, the following table may be useful:

A	λ	λ_A	refer to:
R^t	c_0 and c	r_0^t and r_c^t	[1]
C_1	c_0 and c	$\widetilde{c}_0$ and $\widetilde{c}$	[30]
A^r	c_0 and c	a_0^r and a_c^r	[5]
E^r	c_0 and c	e_0^r and e_c^r	[4]
Δ	c_0 and c	$c_0(\Delta)$ and $c(\Delta)$	[20]
Δ^2	c_0 and c	$c_0(\Delta^2)$ and $c(\Delta^2)$	[17]
$u\Delta^2$	c_0 and c	$c_0(u;\Delta^2)$ and $c(u;\Delta^2)$	[25]
Δ^m	c_0 and c	$c_0(\Delta^m)$ and $c(\Delta^m)$	[16, 15]
$B(r,s)$	c_0 and c	$\widehat{c}_0$ and $\widehat{c}$	[21]
R^q	c_0 and c	$(\overline{N},q)_0$ and $(\overline{N},q)$	[23]
$\Delta^{(m)}$	c_0 and c	$c_0(\Delta^{(m)})$ and $c(\Delta^{(m)})$	[22]
$G(u,v)$	c_0 and c	$c_0(u,v)$ and $c(u,v)$	[3]
Λ	c_0 and c	c_0^λ and c^λ	[24]
$B(r,s,t)$	c_0 and c	$B(c_0)$ and $B(c)$	[29]
A_λ	c_0 and c	$A_\lambda(c_0)$ and $A_\lambda(c)$	[7]
$B(\widetilde{r},\widetilde{s})$	c_0 and c	$\widetilde{c}_0$ and $\widetilde{c}$	[11]
$\widetilde{\Lambda}$	c_0 and c	$c_0^\lambda(\widetilde{B})$ and $c_0^\lambda(\widetilde{B})$	[12]
$\widehat{F}$	c_0 and c	$c_0(\widehat{F})$ and $c(\widehat{F})$	[10]

Table 1. The domains of some triangle matrices in the spaces c_0 and c.

Acknowledgment

We would like to thank Dr. Medine Yeşilkayagil, Department of Mathematics, Uşak University, 1 Eylül Campus, 64200 – Uşak, Turkey, for her careful reading and valuable suggestions on the earlier version of this paper which improved the presentation and readability.

References

[1] Altay, B., Başar, F. ,(2006), Some paranormed Riesz sequence spaces of non-absolute type, Southeast Asian Bull. Math., 30(5), pp.591–608.

[2] Altay, B., Başar, F., Mursaleen, M.,(2006), On the Euler sequence spaces which include the spaces ℓ_p and ℓ_∞ I, Inform. Sci., 176(10), pp.1450–1462.

[3] Altay, B., Başar, F.,(2006), Some paranormed sequence spaces of non-absolute type derived by weighted mean, J. Math. Anal. Appl., 319(2), pp.494–508.
[4] Altay, B., Başar, F., (2005), On some Euler sequence spaces of non-absolute type, Ukrainian Math. J., 57(1), pp.1-17.
[5] Aydın, C., Başar, F., (2004), On the new sequence spaces which include the spaces c_0 and c,Hokkaido Math. J., 33(2), pp.383–398.
[6] Aydın, C., Başar, F., (2005), Some new sequence spaces which include the spaces ℓ_p and ℓ_∞, Demonstratio Math., 38(3), pp.641–656.
[7] Braha, N.L., Başar, F., (2013), On the domain of the triangle $A(\lambda)$ on the spaces of null, convergent and bounded sequences, Abstr. Appl. Anal., Article ID 476363, 9p., doi:10.1155/2013/476363.
[8] Başar, F., Altay, B., (2003), On the space of sequences of p-bounded variation and related matrix mappings, Ukrainian Math. J., 55(1), pp.136–147.
[9] Başar, F., (2012), Summability Theory and Its Applications, Bentham Science Publishers, e-books, Monographs, Istanbul.
[10] Başarır, M., Başar, F., Kara, E.E., (2013), On the spaces of Fibonacci difference null and convergent sequences, arXiv:1309.0150v1 [math.FA].
[11] Candan, M., (2012), Domain of the double sequential band matrix in the classical sequence spaces, J. Inequal. Appl., 2012:281, 15p.
[12] Candan, M.,(2014), Domain of the double sequential band matrix in the spaces of convergent and null sequences, Adv. Difference Equ., 2014:163, 17p.
[13] Choudhary, B., Nanda, S., (1989), Functional Analysis with Applications, John Wiley and Sons Inc. New Delhi.
[14] Cakmak, A.F., Başar, F., (2015), Some sequence spaces and matrix transformations in multiplicative sense, TWMS J. Pure Appl. Math., 6(1), pp.27-37.
[15] Çolak, R., Et, M., (1997), On some generalized difference sequence spaces and related matrix transformations, Hokkaido Math. J., 26(3), pp.483–492.
[16] Et, M., Çolak, R., (1995), On some generalized difference sequence spaces, Soochow J. Math., 21(4), pp.377–386.
[17] Et, M., (1993), On some difference sequence spaces, Turkish J. Math., 17, pp.18–24.
[18] Hardy, G.H., (1956), Divergent Series, Oxford University Press, London.
[19] Jarrah, A.M., Malkowsky, E., (1990), BK spaces, bases and linear operators, Rendiconti Circ. Mat. Palermo II, 52, pp.177–191.
[20] Kızmaz, H., (1981), On certain sequence spaces, Canad. Math. Bull., 24(2), pp.169–176.
[21] Kirişçi, M., Başar, F., (2010), Some new sequence spaces derived by the domain of generalized difference matrix, Comput. Math. Appl., 60(5), pp.1299–1309.
[22] Malkowsky, E., Parashar, S.D., (1997), Matrix transformations in space of bounded and convergent difference sequences of order m, Analysis, 17, pp.87–97.
[23] Malkowsky, E., (1997), Recent results in the theory of matrix transformations in the sequence spaces, Mat. Vesnik, 49, pp.187–196.
[24] Mursaleen, M., Noman, A.K., (2010), On the spaces of λ-convergent and bounded sequences, Thai J. Math., 8(2), pp.311–329.
[25] Mursaleen, M., (1996), Generalized spaces of difference sequences, J. Math. Anal. Appl., 203(3), pp.738–745.
[26] Mears, F.M., (1935), Some multiplication theorems for the Nörlund mean, Bull. Amer. Math. Soc., 41(12), pp.875–880.
[27] Mears, F.M., (1943), The inverse Nörlund mean, Ann. Math., 44(3), pp.401–409.
[28] Ng, P.-N., Lee, P.-Y., (1978), Cesàro sequence spaces of non–absolute type, Comment. Math. Prace Mat., 20(2), pp.429–433.
[29] Sönmez, A., (2011), Some new sequence spaces derived by the domain of the triple band matrix, Comput. Math. Appl., 62(2), pp.641-650.
[30] Şengönül, M., Başar, F., (2005), Some new Cesàro sequence spaces of non-absolute type which include the spaces c_0 and c, Soochow J. Math., 31(1), pp.107–119.
[31] Wang, C.S., (1978), On Nörlund sequence space, Tamkang J. Math., 9, pp.269–274.
[32] Wilansky, A., (1984), Summability through Functional Analysis, North-Holland Mathematics Studies, 85, Amsterdam-Newyork-Oxford.

Orhan Tuğ is a member of Department of Mathematical Education of Ishik University, Erbil-Iraq and is an assistant lecturer since 2011 and is the head of the Department of Mathematics Education since 2013. His research area is in summability, matrix methods, double and single sequence spaces, Schauder bases, matrix transformations, the alpha-, beta- and gamma-duals, ideal convergence and statistical convergence. His special interests are functional analysis and applied summability methods.

Feyzi Başar, for a photograph and biography, see TWMS J. Pure Appl. Math., V.6, N.1, 2015, p.37